MIDTERM TEST 18-05-2015

ELECTRICITY AND MAGNETISM 1. 09:00-11:00, A. JACOBSHAL 01,
QUESTIONS: 3, # POINTS: 100

Write your name and student number on every sheet. Use a separate sheet for each problem. Write clearly. Use of a (graphing) calculator is allowed. For all problems you have to write down your arguments and the intermediate steps in your calculations.

Question 1 - Spheres (40 points)

A. Show that

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$$

is equivalent to

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

B. Calculate the electric field inside and outside a uniformly charged shell of radius R, with total charge q.

C. Calculate the electric field inside and outside a uniformly charged solid sphere of radius R, with total charge q.

D. Draw the electric field inside a solid sphere with uniform polarisation $\vec{P} = P\hat{z}$ per unit volume, with radius R.

E. For the sphere from question D, draw the electric field outside the sphere.

F. If we were to create a small cavity inside this uniformly polarised sphere, and place a charge -q inside this cavity, what would be (to good approximation) the electric field far away from the sphere ?

Question 2 - Work and potential (30 points)

A. Show that, because $\vec{\nabla} \times \vec{E} = 0$, we can define a scalar function V such that $\vec{E} = -\vec{\nabla}V$.

B. We place three negative charged particles at the corners of a equilateral triangle with all sides a. What is the kinetic energy of the top charge, if we let this one fly away, while the other two charges are kept fixed at their locations?

C. Consider a metal sphere of radius R which carries a charge q. It is surrounded, out to a radius b, by linear dielectric material of permittivity ϵ. Find the potential at the center (relative to infinity).

Question 3 - Capacitance (30 points)

A. Show through calculation that the capacitance of two concentric metal shells with radius a and b approaches the capacitance of two large parallel metal surface plates of area A held a small distance d apart, for the conditions $b - a \approx d$ and $d \ll a$.

The End

THE ANSWERS

QUESTION 1 - SPHERES (40 POINTS)

A. (5) Use the divergence theorem: $\int(\vec{\nabla}\cdot\vec{E})d\tau = \oint \vec{E}\cdot d\vec{a}$.

B. (10) Outside: $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$ through Gauss's law.
Inside: $\vec{E} = 0$ through Gauss's law - there is no enclosed charge.

C. (10) Outside: $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$ through Gauss's law (example 2.3).
Inside: $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{r}$, also from Gauss's law (lecture notes 5)

D. (5) The electric field is also uniform, but pointing in the -z direction (figure 4.10).

E. (5) The electric field outside is like a perfect dipole at the centre of the sphere (figure 4.10).

F. (5) Multipole expansion: the dominant contribution is from the monopole charge, so we have a field far away that is approximated by $\vec{E} = -\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$.

QUESTION 2 - WORK AND POTENTIAL (30 POINTS)

A. (10) Because $\vec{\nabla}\times\vec{E} = 0$, $\oint \vec{E}\cdot d\vec{l} = 0$. Thus we can define a function for the potential difference between two points $V(b) - V(a) = -\int_a^b \vec{E}\cdot d\vec{l}$, which is independent of the path. Through the fundamental theorem for gradients, $V(b) - V(a) = \int_a^b(\vec{\nabla}V)\cdot d\vec{l}$, so $\int_a^b(\vec{\nabla}V)\cdot d\vec{l} = -\int_a^b \vec{E}\cdot d\vec{l}$. Because this holds for any points a and b, we have $\vec{E} = -\vec{\nabla}V$.

B. (5) All potential energy of this charge is converted into kinetic energy. The potential energy is $\frac{1}{4\pi\epsilon_0}\frac{2q^2}{a}$ (there is a contribution of $\frac{q^2}{a}$ from each of the two bottom charges).

C. (15) Example 4.5: First calculate the displacement $\vec{D}$ through $\oint \vec{D}\cdot d\vec{a} = Q_{f_{enc}}$, from which you get $\vec{E} = \vec{D}/\epsilon = \frac{q}{4\pi\epsilon r^2}\hat{r}$ for $R < r < b$, and $\vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}$ for $r > b$. For $r < R$, $\vec{E} = \vec{P} = \vec{D} = 0$. From $V = -\oint \vec{E}\cdot d\vec{l}$ over the three regions, you find for the potential at the center $V = \frac{q}{4\pi}(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon R} - \frac{1}{\epsilon b})$.

QUESTION 3 - CAPACITANCE (30 POINTS)

A. (30) Example 2.11 and 2.12. Put $+Q$ and $-Q$ charge on the plates / shells, and you will find the electric field $E = \frac{Q}{\epsilon_0 A}$ in between the plates, and $E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$ between the shells. You calculate the potential through $V = -\int \vec{E}\cdot d\vec{l}$, and arrive at $C = \frac{Q}{V} = \frac{A\epsilon_0}{d}$ for the plates, and $C = 4\pi\epsilon_0\frac{ab}{b-a}$ for the shells. If you put in the conditions from the question, you see that both give the same result (for $A = 4\pi r^2$).

THE END